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Menger curvature and rectifiability in metric spaces

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Abstract

We show that for any metric space X the condition

$$\int_X \int_X \int_X c(z_1, z_2, z_3)^2 \, d\mathcal{H}^1 z_1 \, d\mathcal{H}^1 z_2 \, d\mathcal{H}^1 z_3 < \infty,$$

where $c(z_1, z_2, z_3)$ is the Menger curvature of the triple (z_1, z_2, z_3) and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure on X, guarantees that X is rectifiable.
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1. Introduction

Throughout the paper (X, d) is a metric space. Let z_1, z_2 and z_3 be three points of X. The *Menger curvature* of the triple (z_1, z_2, z_3) is

$$c(z_1, z_2, z_3) = \frac{2 \sin \sphericalangle z_1 z_2 z_3}{d(z_1, z_3)},$$

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where

$$\sphericalangle z_1z_2z_3 = \arccos\frac{d(z_1,z_2)^2 + d(z_2,z_3)^2 - d(z_1,z_3)^2}{2d(z_1,z_2)d(z_2,z_3)}.$$

Note that $c(z_1,z_2,z_3)$ is the reciprocal of the radius of the circle passing through x_1, x_2 and x_3 whenever $\{x_1,x_2,x_3\} \subset \mathbb{R}^2$ is an isometric triple for $\{z_1,z_2,z_3\}$. For $K \in [1,\infty]$, a Borel subset $Z \subset X$ and a Borel measure μ on X we set

$$c_K^2(Z,\mu) = \int\limits_{\mathcal{T}_K(Z)} c(z_1,z_2,z_3)^2\, d\mu^3(z_1,z_2,z_3),$$

where

$$\mathcal{T}_K(Z) = \big\{(z_1,z_2,z_3) \in Z^3\colon\ d(z_i,z_j) < Kd(z_k,z_l) \text{ for all } i,j,k,l \in \{1,2,3\},\ k \neq l\big\}.$$

We also write $c_K^2(Z) = c_K^2(Z,\mathcal{H}^1)$ and $c^2(Z) = c_\infty^2(Z,\mathcal{H}^1)$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure on X (or equivalently on Z).

The diameter of Z is denoted by $d(Z)$ and $B(x,r)$ stands for the closed ball in X with center $x \in X$ and radius $r > 0$. If $W \subset U \times V$ and $u \in U$, where U and V are any sets, we write $W_u = \{v \in V\colon\ (u,v) \in W\}$. For $U_0 \subset U$, a measure μ on U and a function $f : U_0 \to \overline{\mathbb{R}}$ we use the notation

$$\fint\limits_{U_0} f\, d\mu = \frac{1}{\mu(U_0)}\int\limits_{U_0} f\, d\mu$$

if the right-hand side is defined. We say that a metric space X is *rectifiable* if there is $E \subset \mathbb{R}$ and a Lipschitz function $f : E \to X$ such that $\mathcal{H}^1(X \setminus f(E)) = 0$.

In this paper we will prove the following theorem.

Theorem 1.1. *If X is a metric space with $c^2(X) < \infty$ then X is rectifiable.*

It was already known that any Borel set $X \subset \mathbb{R}^n$ with $\mathcal{H}^1(X) < \infty$ and $c^2(X) < \infty$ is rectifiable. This was first proved by David in an unpublished paper. In [4] Léger gave a different proof. Further a very different proof in the case $n = 2$ has been given by Tolsa in [8]. The proof of Theorem 1.1 given here follows the ideas of David's proof. As a matter of fact, the basic idea and some parts of our proof are taken quite directly from it. This result was a part of the argument, when David proved in [1] that a purely unrectifiable set in $\mathbb{C}$ with finite length measure is removable for bounded analytic functions. Under the additional assumption that the set is 1-Ahlfors-regular this was already proved by Mattila, Melnikov and Verdera in [5]. Also they used the curvature by showing that $E \subset \mathbb{C}$ is contained in an Ahlfors-regular curve if there is $C < \infty$ such that $c^2(E \cap D) \leqslant Cd(D)$ for every disc D in $\mathbb{C}$. In [3] we showed that for a bounded 1-Ahlfors-regular metric space X the condition $c_K^2(X) < \infty$, where K is a universal constant large enough, implies that X is a Lipschitz image of a bounded subset of $\mathbb{R}$. More precisely, in this case one can find $E \subset [0,1]$ and a Lipschitz surjection $f : E \to X$ with Lipschitz constant less than $C(c_K^2(X) + d(X))$, where the constant C depends only on the 1-Ahlfors-regularity constant of X. Recall that the 1-Ahlfors-regularity of X means that there exists a constant $C < \infty$ such that $C^{-1}r \leqslant \mathcal{H}^1(B(x,r)) \leqslant Cr$ whenever $x \in X$ and $r \in]0, d(X)]$.

Most of this article will be spent on proving the following proposition.

Proposition 1.2. *For any positive numbers μ_0, C_0 and τ_0 there exist $K < \infty$ and $\varepsilon_0 > 0$ such that if X is a separable metric space and μ is a Borel measure on X verifying*

(i) $\mu(X) \geqslant \mu_0 d(X)$,
(ii) $\mu(B(x,r)) \leqslant C_0 r$ *for any $x \in X$ and $r > 0$,*
(iii) $c_K^2(X,\mu) \leqslant \varepsilon_0 d(X)$,

then there is $E \subset [0,1]$ and a Lipschitz function $f : E \to X$ such that the Lipschitz constant of f is at most $(1+\tau_0)d(X)$ and $\mu(X \setminus f(E)) \leqslant \tau_0 d(X)$.

For any $\phi \in [0,1]$ we denote by $\mathcal{O}(\phi)$ the set of the metric spaces X for which $d(x,z) \geqslant d(x,y) + \phi d(y,z)$ whenever $x,y,z \in X$ are such that $d(x,z) = d(\{x,y,z\})$. Notice that $\{x,y,z\} \in \mathcal{O}(\phi)$ whenever $\cos \sphericalangle xyz \leqslant -\phi \leqslant 0$. We say that a metric space X *is orderable*, if there is an injection $o : X \to \mathbb{R}$ such that for all $x,y,z \in X$ the condition $o(x) < o(y) < o(z)$ implies $d(x,z) > \max\{d(x,y), d(y,z)\}$. In that case the function o is called an *order*. If there is an order o on $\{x_1,\dots,x_m\}$, $m \in \mathbb{N}$, such that $o(x_i) < o(x_{i+1})$ for all $i \in \{1,\dots,m-1\}$, we write shortly $x_1 x_2 \dots x_m$. We also denote $\mathcal{O}_o(\phi) = \{X \in \mathcal{O}(\phi)\colon X \text{ is orderable}\}$. The proof of the next lemma can be found in [2].

Lemma 1.3. *For any $L \geqslant 1$ there is $\phi < 1$ such that if $Z \in \mathcal{O}(\phi)$ and $d(x,y) < Ld(z,w)$ for all $x,y,z,w \in Z$, $z \neq w$, then Z is orderable or $Z = \{v_1, v_2, v_3, v_4\}$ with $v_1v_2v_3$, $v_2v_3v_4$, $v_3v_4v_1$ and $v_4v_1v_2$.*

The following very simple lemma (see [3]) will also be used later.

Lemma 1.4. *Let $\{x,y,z,z_1\}$ be a metric space such that $\{x,y,z\}, \{x,y,z_1\} \in \mathcal{O}(\phi)$.*

(i) *If xyz and $d(z,z_1) < \phi \min\{d(x,y), d(y,z) + d(y,z_1)\}$, then xyz_1.*
(ii) *If xzy and $d(z,z_1) < \phi \min\{d(z,x) + d(z_1,x), d(z,y) + d(z_1,y)\}$, then xz_1y.*

The next lemma is very useful in the proof of Proposition 1.2.

Lemma 1.5. *For any $\eta > 0$ there are positive numbers η_1 and η_2 such that the following is true: Let X be a metric space, μ a Borel measure on X, $\delta \in [0,\infty[$ and $r \geqslant d(X)$. If $\mu(X) \geqslant \delta r$ and $c_5^2(X,\mu) \leqslant \eta_1 \delta^3 r$, then $\mu(B(x,\eta r)) \geqslant \eta_2 \mu(X)$ for some $x \in X$.*

Proof. Fix $K = 5$ and let $\phi \in]3/5, 1[$ be some fixed constant. We assume that $r\mu(X) \in]0,\infty[$. The case $\mu(X) = \infty$ can be treated similarly. Choose $u_1 \in X$ such that

$$\mu(X) \int\limits_{\mathcal{T}_K(X)_{u_1}} c(u_1,y,z)^2 \, d\mu^2(y,z) \leqslant c_K^2(X,\mu),$$

and set $A_k = B(u_1, r\lambda^{k-1}) \backslash B(u_1, r\lambda^k)$ for $k \in \mathbb{N}$, where $\lambda \in](2\phi)^{-1}, (K-2)(K\phi)^{-1}]$. Further let k_0 be the smallest integer such that $\lambda^{k_0} \leqslant a$, where $a = 1 - \phi\lambda \in [2/K, 1/2[$. We first show that there exists $Z \subset X$ such that $d(Z) \leqslant 2ar$ and $\mu(Z) \geqslant (3k_0+1)^{-1}\mu(X)$.

Let us denote $b = (3k_0+1)^{-1}$ and assume that $\mu(B(u_1, ar)) < b\mu(X)$. Then $\mu(A_k) \geqslant 3b\mu(X)$ for some $k \in \{1, 2, \dots, k_0\}$. We now choose $u_2 \in A_k$ such that

$$\mu(X)\mu(A_k) \int\limits_{\mathcal{T}_K(X)_{(u_1,u_2)}} c(u_1, u_2, z)^2 \, d\mu z \leqslant 2c_K^2(X, \mu),$$

$$\mu(A_k) \int\limits_{\mathcal{T}_K(X)_{u_2}} c(x, u_2, z)^2 \, d\mu x \, d\mu z \leqslant 2c_K^2(X, \mu). \tag{1}$$

We can assume that $\mu(A_k \backslash B(u_2, ar)) \geqslant 2b\mu(X)$. Since $Ka \geqslant 2$, we can choose $u_3 \in A_k \backslash B(u_2, ar)$ such that

$$\mu(X)\mu\bigl(A_k \backslash B(u_2, ar)\bigr) \int\limits_{\mathcal{T}_K(X)_{(u_1,u_3)}} c(u_1, y, u_3)^2 \, d\mu y \leqslant 3c_K^2(X, \mu), \tag{2}$$

$$\mu(A_k)\mu\bigl(A_k \backslash B(u_2, ar)\bigr) \int\limits_{\mathcal{T}_K(X)_{(u_2,u_3)}} c(x, u_2, u_3)^2 \, d\mu x \leqslant 6c_K^2(X, \mu), \tag{3}$$

$$\mu(X)\mu(A_k)\mu\bigl(A_k \backslash B(u_2, ar)\bigr)c(u_1, u_2, u_3)^2 \leqslant 6c_K^2(X, \mu). \tag{4}$$

Denote $F = \{w \in A_k\colon \{w, u_1, u_2, u_3\} \in \mathcal{O}(\phi)\}$. We next show that $F \subset B(u_2, ar) \cup B(u_3, ar)$. For this, assume that $w_1, w_2, w_3 \in A_k$ are distinct points such that $\{w_1, w_2, w_3, u_1\} \in \mathcal{O}(\phi)$, and denote $d_i = d(w_i, u_1)$ and $d_{ij} = d(w_i, w_j)$ for $i, j \in \{1, 2, 3\}$. Now $d(w_i, u_1) = d(\{w_i, w_j, u_1\})$ for some distinct $i, j \in \{1, 2, 3\}$, because else we would have, by assuming $d_1 \leqslant d_2 \leqslant d_3$, that $d_{pq} + \phi d_{qr} - d_{pr} \geqslant d_2 + \phi d_1 + \phi(d_3 + \phi d_1) - d_2 - d_3 = (\phi - 1)d_3 + \phi(1+\phi)d_1 > (\phi - 1 + \phi(1+\phi)\lambda)r\lambda^{k-1} \geqslant 0$ for every distinct $p, q, r \in \{1, 2, 3\}$, which is a contradiction. Further, if $d(w_i, u_1) = d(\{w_i, w_j, u_1\})$ then $d_{ij} < (1 - \phi\lambda)r\lambda^{k-1} \leqslant ar$. Therefore, since $d(u_2, u_3) > ar$, we have $F \subset B(u_2, ar) \cup B(u_3, ar)$.

Since $b^2r^2c(u_1, u_2, u_3)^2 \leqslant \eta_1$ by (4), we have $\{u_1, u_2, u_3\} \in \mathcal{O}(\sqrt{1 - 4^{-1}b^{-2}\eta_1}) \subset \mathcal{O}(\phi)$ by assuming $\eta_1 \leqslant 3b^2(1-\phi^2)$. Thus $A_k \backslash (B(u_2, ar) \cup B(u_3, ar)) \subset F_{12} \cup F_{13} \cup F_{23}$, where $F_{ij} = \{w \in A_k \backslash (B(u_2, ar) \cup B(u_3, ar))\colon \{u_i, u_j, w\} \notin \mathcal{O}(\phi)\}$. Now by (1)

$$\mu(F_{12}) \leqslant \int\limits_{F_{12}} \frac{r^2 c(u_1, u_2, w)^2}{4(1-\phi^2)} \, d\mu w \leqslant \frac{\eta_1 \delta r}{6b(1-\phi^2)}$$

and from (2) and (3) we similarly get $\mu(F_{13}) \leqslant 3\eta_1\delta r(8b(1-\phi^2))^{-1}$ and $\mu(F_{23}) \leqslant \eta_1\delta r(4b^2(1-\phi^2))^{-1}$. Thus by taking $\eta_1 \leqslant b^3(1-\phi^2)$ we have

$$\mu\bigl(A_k \backslash \bigl(B(u_2, ar) \cup B(u_3, ar)\bigr)\bigr) < \frac{\eta_1 \delta r}{b^2(1-\phi^2)} \leqslant b\delta r \leqslant b\mu(X)$$

and further $\max\{\mu(B(u_2, ar)), \mu(B(u_3, ar))\} \geqslant b\mu(X)$.

The desired result now follows easily by iterating the above routine. Namely, we can choose $\eta_1 = b^{3N}(2a)^{2-2N}(1-\phi^2)$, where N is the smallest positive integer such that $(2a)^N \leqslant 2\eta$. Then by the above calculation we find inductively $x_n \in X$, $n = 1, \dots, N$, such that $\mu(B(x_n, a(2a)^{n-1}r)) \geqslant b^n \delta r$ for each n. $\square$

Denote

$$\partial(z_1, z_2, z_3) = \min_{\sigma \in S_3} \bigl(d(z_{\sigma(1)}, z_{\sigma(2)}) + d(z_{\sigma(2)}, z_{\sigma(3)}) - d(z_{\sigma(1)}, z_{\sigma(3)})\bigr),$$

where S_3 is the set of permutations on $\{1, 2, 3\}$. For Borel subset $Z \subset X$ we set

$$\beta(Z) = \int\limits_{Z^3} \frac{\partial(z_1, z_2, z_3)}{d(\{z_1, z_2, z_3\})^3}\, d\bigl(\mathcal{H}^1\bigr)^3(z_1, z_2, z_3).$$

One easily sees [2, Lemma 5.1] that for any $K \in [1, \infty[$

$$\frac{c_K^2(Z)}{4K^2} \leqslant \beta(Z) \leqslant \frac{c^2(Z)}{2}. \tag{5}$$

Lemma 1.6. *If X is a metric space with $\beta(X) < \infty$ then the 1-dimensional Hausdorff measure on X is σ-finite.*

Proof. We can assume that X is bounded. As in the proof of the previous lemma, we find $x_0 \in X$ such that for any $\lambda \in]2^{-1/2}, 1[$ and $k \in \mathbb{N}$ there are Borel sets $F^1_{\lambda,k}$, $F^2_{\lambda,k}$ and $F^3_{\lambda,k}$ such that $B(x_0, \lambda^{k-1} d(X)) \backslash B(x_0, \lambda^k d(X)) = F^1_{\lambda,k} \cup F^2_{\lambda,k} \cup F^3_{\lambda,k}$, where $d(F^1_{\lambda,k}), d(F^2_{\lambda,k}) \leqslant 2(1-\lambda^2)\lambda^{k-1} d(X)$ and $\mathcal{H}^1(F^3_{\lambda,k}) < \infty$. Taking a sequence $\lambda_j \uparrow 1$ we have

$$F := X \Bigg\backslash \Bigg(\{x_0\} \cup \bigcup_{j,k \in \mathbb{N}} F^3_{\lambda_j,k}\Bigg) \subset \bigcup_{k \in \mathbb{N}} F^1_{\lambda_i,k} \cup \bigcup_{k \in \mathbb{N}} F^2_{\lambda_i,k}$$

for all $i \in \mathbb{N}$. Since now $1 - \lambda^2 \to 0$ and

$$\bigl(1-\lambda^2\bigr) \sum_{k=1}^{\infty} \lambda^{k-1} = 1 + \lambda \to 2$$

as $\lambda \uparrow 1$, we have $\mathcal{H}^1(F) \leqslant 8d(X) < \infty$. $\square$

By (5) the following theorem implies Theorem 1.1.

Theorem 1.7. *If X is a metric space with $\beta(X) < \infty$ then X is rectifiable.*

A minor modification of the following lemma can be found in [4] where it is stated for a set in $\mathbb{R}^n$, but the proof, which uses the density theorem and the Vitali covering theorem for Hausdorff measures, works for any metric space.

Lemma 1.8. *Let X be a metric space with $0 < \mathcal{H}^1(X) < \infty$ and $\beta(X) < \infty$. Then for all $\varepsilon > 0$ there is a Borel set $Z \subset X$ such that*

(i) $\mathcal{H}^1(Z) > d(Z)/40$,
(ii) $\mathcal{H}^1(Z \cap B(z,r)) \leqslant 3r$ *for any $z \in Z$ and $r > 0$,*
(iii) $\beta(Z) \leqslant \varepsilon d(Z)$.

Taking Proposition 1.2 for granted we can now give a proof of Theorem 1.7 following [4].

Proof of Theorem 1.7. Let X be a metric space with $\beta(X) < \infty$. By Lemma 1.6 we may assume $\mathcal{H}^1(X) < \infty$. Suppose to the contrary that X is not rectifiable. Then there is a subset $Y \subset X$ such that $\mathcal{H}^1(Y) > 0$ and $\mathcal{H}^1(Y \cap g(E)) = 0$ for each Lipschitz function $g : E \to X$ with $E \subset \mathbb{R}$. Let ε_0 and K be as in Proposition 1.2 depending on $\mu_0 = 1/40$, $C_0 = 3$ and $\tau_0 = 1/80$. By Lemma 1.8 we find $Z \subset Y$ so that $\beta(Z) \leqslant \varepsilon_0 d(Z)/4K^2$, $\mathcal{H}^1(Z) > d(Z)/40$ and $\mathcal{H}^1(Z \cap B(z,r)) \leqslant 3r$ for all $z \in Z$ and $r > 0$. Now $c_K^2(Z) \leqslant 4K^2\beta(Z) \leqslant \varepsilon_0 d(Z)$ by (5). Thus by Proposition 1.2 we find a Lipschitz function $f : E \to Z$ such that $E \subset [0,1]$ and $\mathcal{H}^1(Z \setminus f(E)) \leqslant d(Z)/80$. Hence $\mathcal{H}^1(Z \cap f(E)) \geqslant d(Z)/80 > 0$, which is a contradiction. □

One can trivially replace $\beta(X)$ in Lemma 1.8 by the integral $\int_{X^3} g\, d(\mathcal{H}^1)^3$ where $g : X^3 \to [0,\infty]$ is any Borel function. Hence also the condition $c_K^2(X) < \infty$ and $\mathcal{H}^1(X) < \infty$ implies the rectifiability of X provided that the constant K is large enough.

We would like to say something about the converse results in general metric spaces. The following theorem of Schul can be found in [6].

Theorem 1.9. *(See [6].) Let X be a connected 1-Ahlfors-regular metric space. Then*

$$\beta(X) \leqslant C\mathcal{H}^1(X),$$

where the constant C depends only on the 1-Ahlfors-regularity constant of X.

Combining Theorem 1.7 with Theorem 1.9 and [7, Theorem 1.1] one obtains the following characterization of rectifiability.

Corollary 1.10. *A metric space X is rectifiable if and only if X can be written as*

$$X = \bigcup_{i=1}^{\infty} X_i \quad \text{with } \beta(X_i) < \infty \text{ for all } i.$$

2. Preliminaries of the proof of Proposition 1.2

From now on we assume that the hypotheses of Proposition 1.2 are satisfied. Clearly we can assume that $0 < d(X) < \infty$, since else the statement of Proposition 1.2 is trivial. By replacing X by $\varphi(X)$, where $\varphi : X \to \ell^\infty(X)$ is the Kuratowski embedding, we can assume that X is a subset of a Banach space $\mathcal{N}$. We will construct a sequence of curves Γ_n in $\mathcal{N}$ which approximate X. Each Γ_n will be obtained by choosing points $x_i \in X$, $i = 1, 2, \ldots, k(n) \in \mathbb{N}$, and joining x_i to

x_{i+1} by a line segment in $\mathcal{N}$ for each $i \in \{1, 2, \dots, k(n) - 1\}$. We then show that the length of the curve Γ_n

$$l(\Gamma_n) := \sum_{i=1}^{k(n)-1} d(x_i, x_{i+1})$$

is uniformly bounded by $Ld(X)$, where $L < \infty$ is a constant depending on μ_0, ε_0 and C_0. In other words, we find a sequence of $Ld(X)$-Lipschitz surjections $f_n : [0,1] \to \Gamma_n$. Since in our construction the closure of $\bigcup_n \Gamma_n$ is a compact subset of $\mathcal{N}$, we find by the Ascoli–Arzela theorem a $Ld(X)$-Lipschitz function $f : [0,1] \to \mathcal{N}$, which is the uniform limit of some subsequence of (f_n). Finally we show that $\mu(X \setminus \Gamma)$ is small. Here we denote $\Gamma = f([0,1])$. In this section we will construct the sets of the vertices for the approximative curves and prove two lemmas concerning these finite point sets. In Section 3 we give the full construction of the curve $\Gamma = f([0,1])$ and show that its length is bounded by $(1 + \tau_0)d(X)$. In Section 4 we show that $\mu(X \setminus \Gamma) \leqslant \tau_0 d(X)$.

We now describe how we choose the vertices for the curves Γ_n. Let n_0 be the largest integer such that $d(X) \leqslant 2^{-n_0}$, and set $H_{n_0} = D_{n_0-1} = \emptyset$. Let now $n \geqslant n_0$ and assume by induction that we have defined H_n and D_{n-1}. Denote

$$\mathcal{D}_n = \bigcap_{m=n_0}^{n+N_0} \bigl\{x \in X\colon\ \mu\bigl(B\bigl(x, 2^{-m}\bigr)\bigr) \geqslant \delta 2^{-m}\bigr\}, \tag{6}$$

where $N_0 \in \mathbb{N}$ and $\delta > 0$ are constants fixed later. For any $x \in \mathcal{D}_n$ we choose a point $q_n(x) \in B(x, 2^{-n-N_0})$ such that

$$\int_{\mathcal{A}_n(x)} c\bigl(z_1, z_2, q_n(x)\bigr)^2\, d\mu^2(z_1, z_2) \leqslant \fint_{B(x,2^{-n-N_0})} \int_{\mathcal{A}_n(x)} c(z_1, z_2, z_3)^2\, d\mu^2(z_1, z_2)\, d\mu z_3, \tag{7}$$

where $\mathcal{A}_n(x) = \{(z_1, z_2) \in (B(x, R_1 2^{-n}) \setminus B(x, r_1 2^{-n}))^2\colon\ d(z_1, z_2) > r_1 2^{-n}\}$. Here R_1 and r_1 are positive constants fixed later. We set

$$D_n = q_n\bigl(D_n'\bigr), \tag{8}$$

where D_n' is a maximal subset of $\mathcal{D}_n \setminus \bigcup_{y \in H_n} B(y)$ such that $d(z_1, z_2) > 2^{-n}$ for any distinct $z_1, z_2 \in D_n'$. For any $y \in H_n$ we write $B(y) = B(q_{m(y)}^{-1}(y), 2^{-m(y)+3})$, where $m(y)$ is the largest integer m such that $y \in D_m$. We further set

$$H_{n+1} = H_n \cup \left\{x \in D_n\colon\ \mu\left(B\bigl(q_n^{-1}(x), 2^{-n+4}\bigr) \setminus \bigcup_{y \in H_n} B(y)\right) \leqslant C_1 \delta 2^{-n}\right\}, \tag{9}$$

where $C_1 < \infty$ is a constant fixed later. Denote $X_n = D_n \cup H_n$. The curve Γ_n is now determined by the set X_n and an order on X_n.

Notice that X_n is a finite subset of X, because $\mu(X) < \infty$ by (ii). Further

$$d(z_1, z_2) > \bigl(1 - 2^{-N_0+1}\bigr)2^{-n} \geqslant 2^{-n-1} \tag{10}$$

for any distinct $z_1, z_2 \in X_n$ for all $n \geqslant n_0$. Since $D'_{n+1} \subset \mathcal{D}_n \setminus \bigcup_{y \in H_n} B(y)$ we trivially have

$$d\big(x, D'_n\big) \leqslant 2^{-n} \quad \text{for all } x \in D'_{n+1}, \tag{11}$$

$$d(x, D_n) \leqslant \big(1 + 2^{-N_0+1}\big) 2^{-n} < 2^{-n+1} \quad \text{for all } x \in D_{n+1}. \tag{12}$$

For any $x \in X$ and $r > 0$ we set

$$c^2(x, r) = \frac{c_K^2(B(x, r), \mu)}{r}.$$

Let $\varepsilon_1 > 0$ and $Z = \{z \in X\colon c^2(x, r) > \varepsilon_1 \text{ for some } r > 0\}$. Let us choose for each $z \in Z$ a number $r(z)$ such that $c^2(z, r(z)) > \varepsilon_1$. Now $Z \subset \bigcup_{z \in Z} B(z, r(z))$. By the $5r$-covering lemma we find a countable set $Z_1 \subset Z$ such that $Z \subset \bigcup_{z \in Z_1} B(z, 5r(z))$ and $B(z_1, r(z_1)) \cap B(z_2, r(z_2)) = \emptyset$ for distinct $z_1, z_2 \in Z_1$, and we get by (iii)

$$\begin{aligned}\mu(Z) &\leqslant \sum_{z \in Z_1} \mu\big(B\big(z, 5r(z)\big)\big) \leqslant 5C_0 \sum_{z \in Z_1} r(z) < \frac{5C_0}{\varepsilon_1} \sum_{z \in Z_1} c_K^2\big(B\big(z, r(z)\big), \mu\big) \\ &\leqslant \frac{5C_0 \varepsilon_0 d(X)}{\varepsilon_1} \leqslant \frac{\tau_0 d(X)}{2}\end{aligned}$$

as long as $\varepsilon_1 \geqslant 10 C_0 \varepsilon_0 \tau_0^{-1}$. Thus we can without loss of generality assume that

$$c^2(x, r) \leqslant \varepsilon_1 \tag{13}$$

for all $x \in X$ and $r > 0$. We will fix the constant ε_1 later.

The function q_n in (8) and the density requirement in (6) ensure that under certain extra assumptions we can control the increment of the length of the curve by a triple integral of Menger curvature over a suitable subset of X^3 when replacing an old vertice by a new one or just adding a new vertice (see Section 3). Since these extra assumptions are not necessarily always satisfied, the following lemma is crucial in controlling the length of Γ (see especially page 1905 and Lemma 3.2). It will be also used in Section 4 (see (36) and Lemma 4.1).

Lemma 2.1. *For each integer $n \geqslant n_0$, $d(x, X_{n+1}) < 2^{-n+5}$ for all $x \in X_n$.*

Proof. Since $H_n \subset H_{n+1} \subset X_{n+1}$ we can assume that $x \in D_n$ and

$$\mu\left(B\big(q_n^{-1}(x), 2^{-n+4}\big) \setminus \bigcup_{y \in H_n} B(y)\right) > C_1 \delta 2^{-n}.$$

By choosing ε_1 small enough depending on N_0 and $C_1\delta$ and then using Lemma 1.5 we find $z \in B(q_n^{-1}(x), 2^{-n+4}) \setminus \bigcup_{y \in H_n} B(y)$ such that

$$\mu\big(B\big(z, 2^{-n-N_0-1}\big)\big) > \eta C_1 \delta 2^{-n}, \tag{14}$$

where $\eta > 0$ depends on N_0.

We next show that $z \in \mathcal{D}_{n+1}$. If $n \leqslant n_0 + N_0 + 6$ this follows directly from (14) provided that C_1 is big enough depending on N_0. Let us now assume that $n > n_0 + N_0 + 6$. We first show that $z \in \mathcal{D}_{n-N_0-6}$. If this is not true then there is an integer $m \in [n_0, n-6]$ such that $\mu(B(z, 2^{-m})) < \delta 2^{-m}$. Using (11) we find $w \in D_{m+5}$ such that

$$d\big(q_n^{-1}(x), q_{m+5}^{-1}(w)\big) \leqslant 2^{-m-4}. \tag{15}$$

Thus $d(z, q_{m+5}^{-1}(w)) \leqslant 2^{-n+4} + 2^{-m-4} \leqslant 2^{-m-1}$ and so

$$\mu\big(B\big(q_{m+5}^{-1}(w), 2^{-m-1}\big)\big) \leqslant \mu\big(B\big(z, 2^{-m}\big)\big) < \delta 2^{-m}.$$

By choosing $C_1 \geqslant 32$ we have that $w \in H_{m+6}$. From this we get $d(q_n^{-1}(x), q_{m+5}^{-1}(w)) > 2^{-m-2}$, which contradicts (15). So we have $z \in \mathcal{D}_{n-N_0-6}$. This and (14) give $z \in \mathcal{D}_{n+1}$ provided that C_1 is big enough depending on N_0.

If $z \notin B(y)$ for all $y \in H_{n+1}$ then $d(x, D_{n+1}) \leqslant d(x,z) + d(z, D_{n+1}) \leqslant 2^{-n+4} + 2^{-n-N_0} + 2^{-n-1} + 2^{-n-1-N_0} < 2^{-n+5}$. Else $z \in B(y)$ for some $y \in H_{n+1} \setminus H_n$ and we get $d(x, H_{n+1}) \leqslant d(x,z) + d(z, H_{n+1}) \leqslant 2^{-n+4} + 2^{-n+3} + 2^{-n-N_0+1} < 2^{-n+5}$. $\square$

Let $n > n_0$. Let us write

$$D_n^* = \big\{x \in D_n \colon d(x, D_{n-1}) \leqslant \vartheta 2^{-n+1}\big\} = \big\{x_n^1, \ldots, x_n^{j_n}\big\},$$

where $j_n = \# D_n^*$ and ϑ is any fixed constant between $1/4$ and $1/3$. We define $X_{n-1}^0 = X_{n-1}$ and inductively

$$X_{n-1}^k = \big(X_{n-1}^{k-1} \setminus \big\{p\big(x_n^k\big)\big\}\big) \cup \big\{x_n^k\big\}$$

for $k = 1, \ldots, j_n$, where $p(x)$ be the unique point in X_{n-1}^{k-1} such that $d(x, p(x)) = d(x, X_{n-1}^{k-1})$. Notice that $p(x) \in D_{n-1}$ for all $x \in D_n^*$, and the mapping $p \colon D_n^* \to D_{n-1}$ is injective by (10). This is because N_0 is chosen to be a large integer. Further we denote $k_n = \# D_n$ and write

$$D_n \setminus D_n^* = \big\{x_n^{j_n+1}, \ldots, x_n^{k_n}\big\},$$

where

$$d\big(x_n^k, X_{n-1}^{k-1}\big) = \max\big\{d\big(x, X_{n-1}^{k-1}\big) \colon x \in X_n\big\}, \tag{16}$$

and

$$X_{n-1}^k = X_{n-1}^{j_n} \cup \big\{x_n^{j_n+1}, \ldots, x_n^k\big\}$$

for $k = j_n + 1, \ldots, k_n$.

For any $n \geqslant n_0$ and $z \in X_n \cup D_{n+1}$ we denote

$$m_n(z) = \begin{cases} m(z) & \text{if } z \in H_n, \\ n & \text{if } z \in D_n \setminus D_{n+1}, \\ n+1 & \text{if } z \in D_{n+1} \end{cases}$$

and

$$B_n(z) = B\big(q_{m_n(z)}^{-1}(z), 2^{-m_n(z)-N_0}\big).$$

Suppose that $n \geqslant n_0$, $k \in \{0, \dots, k_{n+1}\}$ and z_1, z_2 are distinct points in X_n^k. Let also $w_i \in B_n(z_i)$ for $i = 1, 2$. By the construction

$$Q_1^{-1} d(z_1, z_2) < d(w_1, w_2) < Q_1 d(z_1, z_2), \tag{17}$$

where $Q_1 = \vartheta(\vartheta - 2^{-N_0+2})^{-1}$. The next lemma allows us to order the vertices of the approximative curves so that usually the length of the curve increases very little when adding a new vertice (see (H1) in Section 3). The constants M_1 and $\phi_1 \in [0, 1[$ will be fixed later.

Lemma 2.2. *The set $B(x, 2^{-n+M_1+1}) \cap X_n^k$ belongs to $\mathcal{O}_o(\phi_1)$ for all $x \in X$, $n \geqslant n_0$ and $k \in \{0, \dots, k_{n+1}\}$.*

Proof. Let us denote $Z = B(x, 2^{-n+M_1+1}) \cap X_n^k$. Now $\vartheta < 2^n d(z_1, z_2) \leqslant 2^{M_1+2}$ for all distinct $z_1, z_2 \in Z$. Assuming that $\#Z \geqslant 2$ the construction gives that $m_n(z) \geqslant n - M_1 - 2$ for all $z \in Z$ (see (10)).

Choose $z_0 \in Z$ and denote $A = B(q_{m_0}^{-1}(z), 2^{-m_0-2}) \setminus B(q_{m_0}^{-1}(z), \sigma 2^{-m_0})$, where $m_0 = m_n(z_0)$. Now $\mu(A) \geqslant (\delta 2^{-2} - C_0\sigma)2^{-m_0}$. By choosing $\sigma > 0$ small enough depending on δ and C_0 and taking ε_1 small enough depending on δ and N_0, and then using Lemma 1.5 we find $y_0 \in A$ such that $\mu(U_{y_0}) \geqslant \eta\mu(A)$, where $U_{y_0} = B(y_0, 2^{-m_0-N_0}) \cap A$ and $\eta > 0$ depends on N_0. Now $l < 2^n d(z_1, z_2) \leqslant L$ for all distinct $z_1, z_2 \in Z \cup \{y_0\}$, where $l = \min\{2^{-2}(\sigma - 2^{-N_0}), \vartheta - 2^{-2} - 2^{-N_0}\}$ and $L = 2^{M_1+2} + 2^{M_1+1}$. Furthermore, choosing N_0 big enough depending on σ and ϑ, for any $y \in U_{y_0}$ and $w \in B_n(z)$, $z \in Z$,

$$Q_2^{-1} d(y_0, z) < d(y, w) < Q_2 d(y_0, z),$$

where $Q_2 = \max\{(\sigma + 2^{-N_0+1})(\sigma - 2^{-N_0})^{-1}, (\vartheta - 2^{-2} - 2^{-N_0})(\vartheta - 2^{-2} - 2^{-N_0+1})^{-1}\}$.

Suppose now to the contrary that $Z \cup \{y_0\} \supset \{z_1, z_2, z_3\} \notin \mathcal{O}(\phi_1)$. For $i = 1, 2, 3$ let $w_i \in U_{y_0}$ if $z_i = y_0$ and $w_i \in B_n(z_i)$ if $z_i \neq y_0$. Denote $d_{ij} = d(z_i, z_j)$ and $d'_{ij} = d(w_i, w_j)$ for $i, j = 1, 2, 3$, and assume that $d(w_1, w_3) = d(\{w_1, w_2, w_3\})$ and $d_{12} \geqslant d_{23}$. By denoting $Q = \max\{Q_1, Q_2\}$ (see (17)) and choosing N_0 big enough depending on σ, ϑ, M_1 and ϕ_1,

$$\frac{d'_{13} - d'_{12}}{d'_{23}} \leqslant \frac{Qd_{13} - Q^{-1}d_{12}}{Q^{-1}d_{23}} \leqslant \frac{d_{13} - d_{12} + (Q^2 - 1)d_{13}}{d_{23}} < \phi_1 + \big(Q^2 - 1\big)L/l < 1.$$

So we have

$$c(z_1, z_2, z_3)^2 = \frac{(2\sin\alpha)^2}{d(z_1, z_3)^2} \geqslant \frac{4(1 - \cos^2\alpha)}{(2^{M_1+3})^2 2^{-2n}} \geqslant \frac{1 - \max\{\theta^2, 1/4\}}{(2^{M_1+2})^2 2^{-2n}},$$

where $\alpha = \sphericalangle z_1 z_2 z_3$ and $\theta = \phi_1 + (Q^2 - 1)L/l$.

By using the above estimates and choosing the constant K big enough depending on l, L and Q we deduce that the number $c^2(x, 2^{-n+M_1+2})$ is larger than some positive constant depending on θ, M_1, δ and N_0. Taking ε_1 small enough this contradicts (13) and so $Z \cup \{y\} \in \mathcal{O}(\phi_1)$. By

Lemma 1.3 we can choose $\phi_1 < 1$ depending only on L/l such that Z is orderable. Notice that if $\#Z = 4$ then we can apply Lemma 1.3 for $Z \cup \{y_0\}$. $\square$

3. Construction and length of Γ

Notice that $\mathcal{D}_{n_0} \neq \emptyset$ by Lemma 1.5 provided that ε_0 and δ are small enough depending on N_0 and μ_0. Thus D_{n_0} consists of one point. Further $X_n \neq \emptyset$ for all integers $n \geqslant n_0$ by Lemma 2.1. We define $\Gamma_{n_0}^0 = D_{n_0}$ and $\mathcal{E}_{n_0}^0 = \emptyset$. For any indices $n \geqslant n_0$ and $k \in \{0, \dots, k_{n+1}\}$ we will denote by $\mathcal{E}_n^k$ the set of the edges of the curve Γ_n^k. So Γ_n^k is determined by $\mathcal{E}_n^k$ unless $\mathcal{E}_n^k$ is empty and Γ_n^k is reduced to one point. We will also write for $y \in X_n^k$

$$N_n^k(y) = \{w \in X_n^k\colon \{y, w\} \in \mathcal{E}_n^k\}.$$

Let now $n \geqslant n_0$ and $k \in \{0, \dots, j_{n+1} - 1\}$, and assume by induction that we have already constructed a curve Γ_n^k such that $X_n^k \subset \Gamma_n^k$ and the following hypothesis is satisfied:

(H1) If $z \in X_n$, $B(z, 2^{-n+M_1}) \cap X_n = \{x_1, \dots, x_j\}$ and $x_1x_2\dots x_j$, then $\{x_i, x_{i+1}\} \in \mathcal{E}_n^0$ for all $i \in \{1, \dots, j-1\}$.

Notice that (H1) is trivially true for $n = n_0$. We now construct a curve Γ_n^{k+1} such that $X_n^{k+1} \subset \Gamma_n^{k+1}$. Denote $x = x_{n+1}^{k+1}$ and let $y \in X_n$ be such that $d(x, y) = d(x, X_n)$. We simply replace y by x, i.e. we set

$$\mathcal{E}_n^{k+1} = \left(\mathcal{E}_n^k \setminus \{\{y, w\}\colon w \in N_n^k(y)\}\right) \cup \{\{x, w\}\colon w \in N_n^k(y)\}.$$

Since $d(z_1, z_2) > (1 - 2^{-N_0+1} - 2\vartheta)2^{-n}$ for any distinct $z_1, z_2 \in X_n^k$ by (10) and $d(x, y) \leqslant \vartheta 2^{-n} \leqslant \phi_1(1 - 2^{-N_0+1} - 2\vartheta)2^{-n}$ by our choice of the constants, we easily see by Lemmas 2.2, 1.4 and (H1) that the following hypothesis will be satisfied for each $j \in \{0, \dots, j_{n+1}\}$:

(H2) If $z \in X_n^j$, $B(z, (2^{M_1} - 1)2^{-n}) \cap X_n^j = \{y_1, \dots, y_l\}$ and $y_1y_2\dots y_l$, then $\{y_i, y_{i+1}\} \in \mathcal{E}_n^j$ for all $i \in \{1, \dots, l-1\}$.

We now want to estimate the difference $l(\Gamma_n^{k+1}) - l(\Gamma_n^k)$ in certain cases. If $\#N_n^k(y) = 1$ we will use the simple estimate

$$l(\Gamma_n^{k+1}) - l(\Gamma_n^k) \leqslant d(x, y) \leqslant \vartheta 2^{-n}. \tag{18}$$

Let us now assume that $\#N_n^k(y) = 2$ and $d(x, w_i) \leqslant 2^{-n+M_2}$ for $i = 1, 2$, where $\{w_1, w_2\} = N_n^k(y)$ and M_2 is a large constant fixed later. Denote $Z = Z(x) \cap Z(w_1) \cap Z(w_2)$, where

$$Z(z) = X_{n+N_1} \cap B(x, 2^{-n+M_2+3}) \setminus B(z, 2^{-n-N_1}).$$

Here N_1 is an integer larger than 10. By the construction $m_{n+N_1}(v) \geqslant n - M_2$ for any $v \in Z$ as well as $m_n(w_i) \geqslant n - M_2$ for $i = 1, 2$. Thus by choosing N_0, K and $\phi_1 < 1$ big enough and ε_1 small enough depending also on N_1 we see as in the proof of Lemma 2.2 that $Z \cup \{x, w_1, w_2\} \in \mathcal{O}_o(\phi_1)$.

Assume first that there does not exist $v \in Z$ such that xvw_1. In this case we content ourselves with showing that an endpoint of $\Gamma^0_{n+N_1}$ (i.e. $z \in X_{n+N_1}$ with $\#N^0_{n+N_1}(z) \leqslant 1$) or a relatively long edge of $\Gamma^0_{n+N_1}$ lies close to x. By Lemma 2.1 we find $u \in X_{n+N_1}$ with $d(x,u) < 2^{-n+5}$. Further we assume that there exist $u_1, u_2 \in X_{n+N_1}$ such that $\{u_1, u, u_2\} \in \mathcal{O}(\phi_1)$, u_1uu_2 and $2 < 2^{n-M_2}d(u,u_i) \leqslant 4$ for $i = 1,2$. If this is not the case then by Lemma 2.2 and (H1) an endpoint of $\Gamma^0_{n+N_1}$ lies in $B(u, 2^{-n+M_2+1}) \subset B(x, 2^{-n+M_2+2})$ or there exists $\{y_1, y_2\} \in \mathcal{E}^0_{n+N_1}$ such that $d(x,y_1) \leqslant 2^{-n+M_2+1} < d(y_1,y_2)$. For this we choose $M_1 \geqslant N_1 + M_2 + 2$. Notice that we used (H1) for $n + N_1$ though we have not verified it yet. This will be done later. Now $u_1, u_2 \in Z$ since $(2^{M_2}+1)2^{-n} \leqslant (2^{M_2+1} - 32)2^{-n} < d(u_i,u) - d(u,x) \leqslant d(u_i,x) \leqslant d(u_i,u) + d(u,x) < (2^{M_2+2} + 32)2^{-n} \leqslant 2^{-n+M_2+3}$ and $d(u_i,w_j) \geqslant d(u_i,u) - d(x,w_j) - d(x,u) > (2^{M_2} - 32)2^{-n} \geqslant 2^{-n-N_1}$ for $i,j = 1,2$. Since $16 \leqslant \phi_1 2^{M_2}$, Lemma 1.4(ii) gives u_1xu_2. Thus $u_1w_1xu_2$ or $u_2w_1xu_1$. Let us choose the indices such that $u_1w_1xu_2$. Denote $Z_1 = \{v \in Z\colon vw_1x\}$ and $Z_2 = \{v \in Z\colon w_1xv\}$. Now $Z = Z_1 \cup Z_2$ and recalling that w_1xw_2 by (H2) (and Lemma 2.2) we have

$$\begin{aligned} d\big(Z_1 \cup B\big(w_1, 2^{-n-N_1}\big), Z_2 \cup B\big(x, 2^{-n-N_1}\big) \cup B\big(w_2, 2^{-n-N_1}\big)\big) &> d(w_1,x) - 2^{-n-N_1+1} \\ &> 2^{-n-3} \end{aligned}$$

by (10). Since $u_i \in Z_i \cap B(u, 2^{-n+M_2+2}) \subset B(x, 2^{-n+M_2+3})$ for $i = 1,2$, we deduce by (H1) (which is not proved yet) that there exists $\{y_1,y_2\} \in \mathcal{E}^0_{n+N_1}$ such that $d(y_1,y_2) > 2^{-n-3}$ and $d(x,y_1) < 2^{-n+M_2+3}$.

We now assume that there exist $v_1, v_2 \in Z$ such that xv_1w_1 and xv_2w_2. Recall that we still assume $\{w_1,w_2\} = N^k_n(y) \subset B(x, 2^{-n+M_2})$ and $w_1 \neq w_2$. We may also suppose that there exists $v_3 \in Z \setminus B(x, 2^{-n+M_2})$. Namely, if such v_3 does not exist then an endpoint of $\Gamma^0_{n+N_1}$ or a relatively long edge of $\Gamma^0_{n+N_1}$ lies close to x as above, which is enough for us for now.

Let us choose v_3w_1x. Denote $\mathcal{A}(z) = \mathcal{A}_{m_n(z)}(q^{-1}_{m_n(z)}(z))$ and $B_i = B_{n+N_1}(v_i)$ for $z = x, w_1, w_2$ and $i = 1,2,3$. Choosing R_1 big enough depending on M_2, and r_1 small enough depending on M_2 and N_1, we have

$$B_i \times B_j \subset \mathcal{A}(x) \cap \mathcal{A}(w_1) \cap \mathcal{A}(w_2)$$

for distinct $i, j \in \{1,2,3\}$. Fix $\lambda > 0$ and let $\mathcal{G} = \mathcal{G}(x) \cup \mathcal{G}(w_1) \cup \mathcal{G}(w_2)$, where

$$\mathcal{G}(z) = \left\{(\zeta,\xi) \in \mathcal{A}(z)\colon c(\zeta,\xi,z)^2 \geqslant G \fint_{\mathcal{A}(z)} c(z_1,z_2,z)^2\, d\mu^2(z_1,z_2) + \lambda\right\}.$$

Here G is a large constant depending on C_0, δ, R_1, N_1 and N_0. By the Tchebychev inequality

$$\begin{aligned} \mu^2(\mathcal{G}) &\leqslant \mu^2\big(\mathcal{G}(x)\big) + \mu^2\big(\mathcal{G}(w_1)\big) + \mu^2\big(\mathcal{G}(w_2)\big) \\ &\leqslant \frac{1}{G}\big(\mu^2\big(\mathcal{A}(x)\big) + \mu^2\big(\mathcal{A}(w_1)\big) + \mu^2\big(\mathcal{A}(w_2)\big)\big) \\ &\leqslant \frac{C_0^2R_1^2(2^{-2} + 2^{2M_2+1})}{G4^n}. \end{aligned} \tag{19}$$

Denote $U_i = \{v \in B_1\colon \{v\} \times B_i \subset \mathcal{G}\}$ for $i = 2, 3$. We next show that there exists $(u_1, u_2, u_3) \in B_1 \times B_2 \times B_3$ such that $(u_1, u_2) \notin \mathcal{G}$ and $(u_1, u_3) \notin \mathcal{G}$. Suppose this is false. Then $B_1 = U_2 \cup U_3$. Letting

$$p(z) = G \fint_{\mathcal{A}(z)} c(z_1, z_2, z)^2 \, d\mu^2(z_1, z_2) + \lambda$$

we have for each $i \in \{2, 3\}$ and $z \in \{x, w_1, w_2\}$

$$\bigl\{v \in B_1\colon \{v\} \times B_i \subset \mathcal{G}(z)\bigr\} = \bigl\{v \in B_1\colon c(v, \xi, z)^2 \geqslant p(z) \text{ for all } \xi \in B_i\bigr\},$$

which is a closed set (in X). Thus U_2 and U_3 are closed and we get

$$\begin{aligned}
\mu^2(\mathcal{G}) &\geqslant \mu^2(U_2 \times B_2) + \mu^2(U_3 \times B_3) \\
&= \mu(U_2)\mu(B_2) + \mu(U_3)\mu(B_3) \\
&\geqslant \bigl(\mu(U_2) + \mu(U_3)\bigr) \min\bigl\{\mu(B_2), \mu(B_3)\bigr\} \\
&\geqslant \mu(B_1) \min\bigl\{\mu(B_2), \mu(B_3)\bigr\} \geqslant \delta^2 4^{-n-N_1-N_0},
\end{aligned}$$

which contradicts (19) provided that G has been chosen big enough.

As in the proof of Lemma 2.2 we easily see that $\{x, w_1, w_2\} \cup V \in \mathcal{O}_o(\phi_1)$ for $V = \{v_1, v_2, v_3\}, \{u_1, u_2, u_3\}$ if we choose ϕ_1 big enough depending on M_2, N_0 big enough depending on M_2, N_1 and ϕ_1, K big enough depending on M_2 and N_1, and ε_1 small enough depending on ϕ_1, M_2, N_0, N_1 and δ. Using $w_1 x w_2$ and the assumptions $x v_1 w_1$, $x v_2 w_2$ and $v_3 w_1 x$ we deduce by Lemma 1.4 that $u_3 w_1 u_1 x u_2 w_2$. Letting

$$\varphi = -\cos\min\{\sphericalangle u_1 x u_2, \sphericalangle u_3 w_1 u_1, \sphericalangle u_1 u_2 w_2, \sphericalangle u_3 u_1 w_2\}$$

we have

$$\begin{aligned}
d(w_1, w_2) &\geqslant d(u_3, w_2) - d(u_3, w_1) \\
&\geqslant \varphi d(u_3, u_1) + d(u_1, w_2) - d(u_3, w_1) \\
&\geqslant \varphi\bigl(\varphi d(u_3, w_1) + d(w_1, u_1)\bigr) + d(u_1, u_2) + \varphi d(u_2, w_2) - d(u_3, w_1) \\
&\geqslant \varphi\bigl(\varphi d(u_3, w_1) + d(w_1, u_1)\bigr) + d(u_1, x) + \varphi d(x, u_2) + \varphi d(u_2, w_2) - d(u_3, w_1) \\
&\geqslant \varphi\bigl(d(w_1, u_1) + d(u_1, x) + d(x, u_2) + d(u_2, w_2)\bigr) + \bigl(\varphi^2 - 1\bigr) d(u_3, w_1) \\
&\geqslant \varphi\bigl(d(w_1, x) + d(x, w_2)\bigr) + \bigl(\varphi^2 - 1\bigr) d(u_3, w_1).
\end{aligned}$$

Denote $\lambda_1 = c(x, u_1, u_2) d(u_1, u_2)$, $\lambda_2 = c(w_1, u_1, u_3) d(u_1, u_3)$, $\lambda_3 = c(w_2, u_1, u_2) d(u_1, w_2)$ and $\lambda_4 = c(w_2, u_1, u_3) d(u_3, w_2)$. Since $4(1 - \varphi^2) = \max\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}^2$, $(u_1, u_2) \notin \mathcal{G}$, $(u_1, u_3) \notin \mathcal{G}$ and $\lambda > 0$ is arbitrary, we further get by (7)

$$\begin{aligned} l(\Gamma_n^{k+1}) - l(\Gamma_n^k) &\leqslant d(w_1, x) + d(x, w_2) - d(w_1, w_2) \\ &\leqslant (1-\varphi)\big(d(w_1, x) + d(x, w_2)\big) + \big(1-\varphi^2\big) d(u_3, w_1) \\ &\leqslant \big(1-\varphi^2\big)\big(d(w_1, x) + d(x, w_2) + d(u_3, w_1)\big) \\ &\leqslant C_2 \int\limits_{B(x, R_2 2^{-n})} \int\limits_{A_n(z_3)} \int\limits_{A_n(z_3)\setminus B(z_2, r_1 2^{-n})} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3, \end{aligned} \tag{20}$$

where $A_n(z) = B(z, R_2 2^{-n}) \setminus B(z, r_1 2^{-n})$, R_2 depends on M_2, and C_2 depends on G and δ.

Let now $k \in \{j_{n+1}, \dots, k_{n+1} - 1\}$ and assume by induction that we have constructed a curve Γ_n^k and the following hypothesis is satisfied:

(H3) If $z \in X_n^{j_{n+1}}$, $B(z, (2^{M_1} - 3)2^{-n}) \cap X_n^k = \{z_1, \dots, z_m\}$ and $z_1 z_2 \dots z_m$, then $\{z_i, z_{i+1}\} \in \mathcal{E}_n^k$ for all $i \in \{1, \dots, m-1\}$.

Clearly (H2) implies (H3) if $k = j_{n+1}$. We again denote $x = x_{n+1}^{k+1}$ and let $y \in X_n^k$ be such that $d(x, y) = d(x, X_n^k)$. Denote $N'(y) = \{w \in N_n^k(y): \ yxw\}$. If $N'(y) \neq \emptyset$, we choose $w \in N'(y)$ such that $d(y, w) = d(y, N'(y))$. If $N'(y) = \emptyset$ and $\#N_n^k(y) = 2$, we choose $w \in N_n^k(y)$ such that $d(y, w) = \max\{d(y, z): \ z \in N_n^k(y)\}$. Then we set

$$\mathcal{E}_n^{k+1} = \big(\mathcal{E}_n^k \setminus \{y, w\}\big) \cup \big\{\{y, x\}, \{x, w\}\big\}.$$

If $N'(y) = \emptyset$ and $\#N_n^k(y) \leqslant 1$, we put

$$\mathcal{E}_n^{k+1} = \mathcal{E}_n^k \cup \big\{\{y, x\}\big\}.$$

Let us note that the construction does not depend on choice of y by (12), Lemma 2.2 and (H3).

If $\#N_n^k(y) \leqslant 1$ and $N'(y) = \emptyset$ we will use the simple estimate

$$l(\Gamma_n^{k+1}) - l(\Gamma_n^k) \leqslant d(x, y) \leqslant \big(1 + 2^{-N_0+1}\big)2^{-n}, \tag{21}$$

which comes from (12).

We next assume that yxw for some $w \in B(y, 2^{-n+M_2}) \cap N_n^k(y)$. Choosing M_1 large enough depending on M_2 and using (12), Lemma 2.2 and (H3) we see that this is the case if $\#N_n^k(y) = 2$ and $N_n^k(y) \subset B(y, 2^{-n+M_2})$. As before, an endpoint of $\Gamma_{n+N_1}^0$ lies in $B(x, 2^{-n+M_2+2})$, or there exists $\{y_1, y_2\} \in \mathcal{E}_{n+N_1}^0$ such that $d(y_1, y_2) > 2^{-n-3}$ and $d(x, y_1) < 2^{-n+M_2+3}$, or

$$l(\Gamma_n^{k+1}) - l(\Gamma_n^k) \leqslant C_2 \int\limits_{B(x, R_2 2^{-n})} \int\limits_{A_n(z_3)} \int\limits_{A_n(z_3)\setminus B(z_2, r_1 2^{-n})} c(z_1, z_2, z_3)^2 \, d\mu z_1 \, d\mu z_2 \, d\mu z_3 \tag{22}$$

as in (20).

Let us next show that (H3) remains valid when we replace k by $k+1$. Assume by induction that also the following condition is satisfied for any $z \in X_n^{j_{n+1}}$. For $k = j_{n+1}$ this follows directly from (H2).

$(*)$ If $w_1 \in B(z, (2^{M_1} - 1)2^{-n}) \cap X_n^{j_{n+1}}$, $\{w_2, \dots, w_{p-1}\} = \{w \in X_n^k\colon\ w_1 w z\}$, $w_p = z$ and $w_1 w_2 \dots w_p$, then $\{w_i, w_{i+1}\} \in \mathcal{E}_n^k$ for all $i \in \{1, \dots, p-1\}$.

We very first show that we can replace k by $k+1$ in $(*)$. Let $z \in X_n^{j_{n+1}}$ and $\{w_1, \dots, w_p\}$ be as in the hypothesis of $(*)$. Clearly we can assume that $w_1 \neq z$. We first notice that $w_1 x z$ implies $y \in \{w_1, \dots, w_p\}$. Namely, $w_1 x z$ implies $d(x, z) \leqslant 2^{-n+M_1}$ and further $d(x, w) > 2^{-n+M_1}$ for all $w \in X_n^k \setminus B(z, 2^{-n+M_1+1})$. Since $B(z, 2^{-n+M_1+1}) \cap X_n^{k+1}$ is orderable by Lemma 2.2, we get the conclusion. Thus we can assume that $y \in \{w_1, \dots, w_p\}$. Since

$$d(x, y) \leqslant d\big(x, X_n^{j_{n+1}}\big) \leqslant \big(1 + 2^{-N_0+1} + \vartheta\big)2^{-n} < 2^{-n+1} \tag{23}$$

by (12), the set $\{x, w_1, \dots, w_p\}$ is orderable by Lemma 2.2. If $y \in \{w_2, \dots, w_{p-1}\}$ then $(*)$ is clearly valid also for $k+1$ by the construction. Assume now that $y = w_1$. By Lemma 2.2 there cannot exist $w \in X_n^k$ such that $y w w_2$. If $y x w_2$ and $\{\{y, x\}, \{x, w_2\}\} \not\subset \mathcal{E}_n^{k+1}$, then by the construction there is $w \in N_n^k(y) \setminus \{w_2\}$ such that $w x y$ and $d(w, y) \leqslant d(y, w_2)$. Thus the quadruple $\{w, y, x, w_2\} \subset B(y, 2^{-n+M_1}) \cap X_n^{k+1}$ is not orderable contradicting Lemma 2.2. If $x y w_2$ and $\{y, w_2\} \notin \mathcal{E}_n^{k+1}$, then by the construction there is $w \in N_n^k(y) \setminus \{w_2\}$ such that $w y x$ and $d(w, y) \leqslant d(y, w_2)$. Thus again $\{w, x, y, w_2\}$ is not orderable. This shows that $(*)$ holds for $k+1$. The case $y = z$ is treated similarly by replacing w_2 by w_{p-1}.

Let us now show that (H3) still holds if we replace k by $k+1$. If $d(y, z) \leqslant (2^{M_1} - 3)2^{-n}$ this can be seen similarly as above. We can clearly assume that $B(z, (2^{M_1} - 3)2^{-n}) \cap \{x, y\} \neq \emptyset$. Let $v \in X_n^{j_{n+1}}$ be such that $d(x, v) = d(x, X_n)$. Then $x, y, v \in B(z, (2^{M_1} - 1)2^{-n})$ and $x, y \in B(v, 2^{-n+2})$ by (23). So we only need consider the case $y \notin Z$, $x \in B(z, (2^{M_1} - 3)2^{-n})$ and $v \neq z \neq y$. Here we let $Z = \{z_1, \dots, z_m\}$ be as in the hypothesis of (H3). Now there do not exist $z', z'' \in Z$ such that $z' x z''$ because $B(z, 2^{-n+M_1}) \cap X_n^{k+1}$ is orderable and $y \notin Z$. Let us choose $y x z_1 z_2 \dots z_m$.

If $v \neq y$ then we must have $y x v z$ or $v y x z$. If $y x v z$ then $y, z_1 \in B(v, 2^{-n+2})$ and we have $\{y, z_1\} \in \mathcal{E}_n^k$ by (H3) for v (and Lemma 2.2). If $v y x z$ or $v = y$ then $\{y, z_1\} \in \mathcal{E}_n^k$ by $(*)$. So in any case $\{y, z_1\} \in \mathcal{E}_n^k$. If $\{x, z_1\} \notin \mathcal{E}_n^{k+1}$ then there is $w \in N_n^k(y) \setminus \{z_1\}$ such that $w x y$ and $d(w, y) \leqslant d(y, z_1)$. Since $x w z_1$ is not possible we must have $w y z_1$. Thus the quadruple $\{w, y, x, z_1\} \subset B(y, 2^{-n+M_1}) \cap X_n^{k+1}$ is contradictingly not orderable and we get (H3) for $k+1$.

Now Γ_{n+1}^0 is obtained simply by removing $D_n \setminus X_{n+1}$ from $X_n^{k_{n+1}}$ so that the order of the points in X_{n+1} does not change. Precisely, denote $D_n \setminus X_{n+1} = \{x_1, \dots, x_m\}$ and set inductively $X_n^{k_{n+1}+i} = X_n^{k_{n+1}+i-1} \setminus \{x_i\}$ for $i = 1, \dots, m$. If $\# N_n^{k_{n+1}+i-1}(x_i) = 2$ we set

$$\mathcal{E}_n^{k_{n+1}+i} = \big(\mathcal{E}_n^{k_{n+1}+i-1} \setminus \big\{\{x_i, w_1\}, \{x_i, w_2\}\big\}\big) \cup \big\{\{w_1, w_2\}\big\},$$

where $\{w_1, w_2\} = N_n^{k_{n+1}+i-1}(x_i)$. If $\# N_n^{k_{n+1}+i-1}(x_i) = 1$ we set

$$\mathcal{E}_n^{k_{n+1}+i} = \mathcal{E}_n^{k_{n+1}+i-1} \setminus \big\{\{x_i, w\}\big\},$$

where $w \in N_n^{k_{n+1}+i-1}(x_i)$. Finally we put $\mathcal{E}_n^0 = \mathcal{E}_n^{k_{n+1}+m}$.

By induction (H3) holds for all $k \in \{j_{n+1}, \dots, k_{n+1}\}$. Clearly we can also replace $X_n^{k_{n+1}}$ by X_{n+1}. For $z \in X_{n+1}$ there is $y \in X_n^{j_{n+1}}$ such that $d(z, y) < 2^{-n+1}$ by (23). Thus, since M_1 is chosen to be a large constant, $B(z, 2^{-n-1+M_1}) \subset B(y, (2^{M_1} - 3)2^{-n})$. So we have (H1) for

$n+1$, and by induction (H1) and the following simple variant of (H3) hold for each integers $n \geqslant n_0$ and $j_{n+1} \leqslant k \leqslant k_{n+1}$. At this time we note that the induction part which started at the beginning of this section is complete and n is not fixed any more.

(H4) If $z \in X_n^k$, $B(z, 2^{-n+M_1-1}) \cap X_n^k = \{z_1, \dots, z_m\}$ and $z_1 z_2 \dots z_m$, then $\{z_i, z_{i+1}\} \in \mathcal{E}_n^k$ for all $i \in \{1, \dots, m-1\}$.

The following lemma and the construction imply that for any $n > n_0$ there is at most two elements k in $\{j_n+1, \dots, k_n\}$ such that x_n^k is an endpoint of the curve Γ_{n-1}^k. In other words, the growth of the curve at its endpoints is very controlled (see (28)).

Lemma 3.1. *Let $n > n_0$ and $j_n < k < m \leqslant k_n$. Assume that $d(x_n^m, x_n^k) = d(x_n^m, X_{n-1}^{m-1})$. Then there is (unique) $z \in B(x_n^k, 2^{-n+3}) \cap N_{n-1}^{m-1}(x_n^k)$ such that $x_n^k x_n^m z$.*

Proof. Let $y_i \in X_{n-1}^{k-1}$ be such that $d(x_n^i, y_i) = d(x_n^i, X_{n-1}^{k-1})$ for $i = k, m$. Now $d(x_n^m, y_m) \leqslant d(x_n^k, y_k) < 2^{-n+2}$ by (16) and (23). The set $\{x_n^k, x_n^m, y_k, y_m\}$ is orderable by Lemma 2.2. By the assumption either $y_m x_n^m x_n^k y_k$ or $x_n^k x_n^m y_k$. Thus the claim follows from Lemma 2.2 and (H4). $\square$

The following lemma says that if any approximative curve has a relatively long edge then each subsequent curve has an edge relatively close to the original one.

Lemma 3.2. *Let $m > n \geqslant n_0$ and $k \in \{0, \dots, k_{n+1}\}$. Assume that $\{x_1, x_2\} \in \mathcal{E}_n^k$ and $d(x_1, x_2) \geqslant 2^{-n+6}$. Then there is $\{y_1, y_2\} \in \mathcal{E}_m^0$ such that $d(x_i, y_i) < 2^{-n+6}$ for $i = 1, 2$.*

Proof. For any $p \geqslant n_0$, $l \in \{0, \dots, k_{p+1}\}$ and $\{z_1, z_2\} \in \mathcal{E}_p^l$ with $d(z_1, z_2) \geqslant 2^{-p+4}$ there is $\{w_1, w_2\} \in \mathcal{E}_p^{k_{p+1}}$ such that $z_i = w_i$ or $w_i \in D_{p+1}$ with $d(z_i, w_i) < 2^{-p+1}$ for $i = 1, 2$. This follows from Lemma 3.1 and the construction. If $d(z_1, z_2) \geqslant 2^{-p+6}$ then by the construction, Lemma 2.1 and (H4) we find $\{y_1, y_2\} \in \mathcal{E}_{p+1}^0$ such that $d(z_i, y_i) < 2^{-p+5}$ and $d(y_1, y_2) \geqslant d(z_1, z_2) - 2^{-p+6}$ for $i = 1, 2$. If $d(z_1, z_2) \leqslant 2^{-p+M_1-1}$ then $d(y_1, y_2) \geqslant d(w_1, w_2)$ by (H4). Thus $d(y_1, y_2) \geqslant 2^{-p+5}$ by choosing $M_1 \geqslant 8$ and the claim follows by induction. $\square$

Let N_2 be a large integer. By choosing $M_1 \geqslant N_2$ we have $X = B(z, 2^{-N_2-n_0+M_1})$ for $z \in X_{N_2+n_0}$, and thus Lemma 2.2 and (H1) imply

$$l\left(\Gamma_{N_2+n_0}^0\right) \leqslant d(X)/\phi_1. \tag{24}$$

For $n > N_2 + n_0$, $k \in \{1, \dots, k_n\}$ we let $p_n^k \in X_{n-1}^{k-1}$ be such that $d(x_n^k, p_n^k) = d(x_n^k, X_{n-1}^{k-1})$. Let now $m > N_2 + n_0$.

Denote

$$\Lambda_n^1 = \left\{k \in \{1, \dots, j_n\}\colon \#N_{n-1}^{k-1}\left(p_n^k\right) = 2 \text{ and } N_{n-1}^{k-1}\left(p_n^k\right) \subset B\left(x_n^k, 2^{-n+M_2+1}\right)\right\},$$
$$\Lambda_n^2 = \left\{k \in \{j_n+1, \dots, k_n\}\colon\ p_n^k x_n^k w \text{ for } w \in B\left(p_n^k, 2^{-n+M_2+1}\right) \cap N_{n-1}^{k-1}(p_n^k)\right\}.$$

Recall that for $k \in \Lambda_n^1 \cup \Lambda_n^2$ an endpoint of $\Gamma_{n-1+N_1}^0$ lies in $B(x_n^k, 2^{-n+M_2+3})$ (see the first term on the right-hand side of inequality (25)), or there exists $\{y_1, y_2\} \in \mathcal{E}_{n-1+N_1}^0$ such that $d(y_1, y_2) >$

2^{-n-2} and $d(x_n^k, y_1) < 2^{-n+M_2+4}$ (cf. term $\lambda_1 l(\Gamma_m^0)$ in (25)), or the estimates (20) and (22) can be used. By (H4) (and (12))

$$\sum_{k\in\Lambda_{n+1}(z,M)} \big(l(\Gamma_n^k) - l\big(\Gamma_n^{k-1}\big)\big) \leqslant \big(\phi_1^{-1} - 1\big)\big(M + 2^{M_2+1}\big)2^{-n}$$

for any $n < m$, $z \in X_m$ and $M \leqslant 2^{M_1-2}$, where

$$\Lambda_n(z, M) = \big\{k \in \Lambda_n^1 \cup \Lambda_n^2\colon\ d\big(x_n^k, z\big) \leqslant M2^{-n}\big\}.$$

Here we use $M_1 \geqslant M_2 + 6$. Using this, Lemmas 3.2 and 2.1, (10), (20) and (22) we get

$$\begin{aligned}
&\sum_{n=N_2+n_0}^{m-1} \sum_{k\in\Lambda_{n+1}^1\cup\Lambda_{n+1}^2} \big(l\big(\Gamma_n^k\big) - l\big(\Gamma_n^{k-1}\big)\big)\\
&\quad\leqslant \sum_{n=N_2+n_0}^{m-1} \big(\phi_1^{-1} - 1\big)\big(2^{M_2+2} + 2^{M_2+1}\big)2^{-n+1} + (\lambda_1 + \lambda_2) l\big(\Gamma_m^0\big)\\
&\qquad + C_2 \sum_{n=N_2+n_0}^{m-1} \sum_{x\in D_{n+1}} \int\limits_{B(x,R_2 2^{-n})} \int\limits_{A_n(z_3)} \int\limits_{A_n(z_3)\setminus B(z_2,r_1 2^{-n})} c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{4(\phi_1^{-1} - 1)(2^{M_2+3} + 2^{-N_1+6} + 2^{M_2+1})}{2^{-3} - 2^{-N_1+7}},\\
\lambda_2 &= \sum_{n=m-N_1+1}^{m-1} \big(\phi_1^{-1} - 1\big)\big(32 + 2^{M_2+1}\big)2^{-n+m+2} < \big(\phi_1^{-1} - 1\big)\big(32 + 2^{M_2+1}\big)2^{N_1+2}.
\end{aligned}$$

By Lemma 2.2 and (10) we have

$$\#\big(B\big(z, M2^{-n}\big) \cap D_{n+1}\big) \leqslant 8M\phi_1^{-1} + 1 \tag{26}$$

for any $n \geqslant n_0$, $z \in X$ and $M \leqslant 2^{M_1+1}$. Furthermore, if $i < j$ and $A_i(z) \cap A_j(z) \neq \emptyset$, then $r_1 2^{-i} < R_2 2^{-j}$ which gives $j - i < (\log R_2 - \log r_1)/\log 2$. Thus by choosing $2^{M_1+1} \geqslant R_2$ and $K \geqslant R_2/R_1 + 1$

$$\begin{aligned}
&\sum_{n=N_2+n_0}^{m-1} \sum_{x\in D_{n+1}} \int\limits_{B(x,R_2 2^{-n})} \int\limits_{A_n(z_3)} \int\limits_{A_n(z_3)\setminus B(z_2,r_1 2^{-n})} c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3\\
&\quad\leqslant C_3 \int\limits_X \sum_{n=N_2+n_0}^{m-1} \int\limits_{A_n(z_3)} \int\limits_{\mathcal{T}_K(X)_{(z_2,z_3)}} c(z_1, z_2, z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3
\end{aligned}$$

$$\leqslant C_3C_4 \int\limits_X \int\limits_X \int\limits_{\mathcal{T}_K(X)_{(z_2,z_3)}} c(z_1,z_2,z_3)^2\, d\mu z_1\, d\mu z_2\, d\mu z_3$$
$$= C_3C_4c_K^2(X,\mu),$$

where $C_3 = 8R_2\phi_1^{-1} + 1$ and $C_4 = (\log R_2 - \log r_1)/\log 2$. Thus by (25) and (iii)

$$\sum_{n=N_2+n_0}^{m-1} \sum_{k\in\Lambda_{n+1}^1\cup\Lambda_{n+1}^2} \bigl(l\bigl(\Gamma_n^k\bigr) - l\bigl(\Gamma_n^{k-1}\bigr)\bigr)$$
$$< \bigl(\phi_1^{-1} - 1\bigr)2^{M_2+5-N_2-n_0} + (\lambda_1+\lambda_2)l\bigl(\Gamma_m^0\bigr) + C_2C_3C_4\varepsilon_0 d(X). \tag{27}$$

By (18), (21) and Lemma 3.1

$$\sum_{n=N_2+n_0}^{m-1} \sum_{k\in\Lambda_{n+1}^3} \bigl(l\bigl(\Gamma_n^k\bigr) - l\bigl(\Gamma_n^{k-1}\bigr)\bigr) \leqslant \sum_{n=N_2+n_0}^{\infty} \bigl(\vartheta + 1 + 2^{-N_0+1}\bigr)2^{-n+1} < 2^{-N_2-n_0+3}, \tag{28}$$

where $\Lambda_n^3 = \{k \in \{1,\dots,k_n\}\setminus\Lambda_n^2\colon \#N_{n-1}^{k-1}(p_n^k) \leqslant 1\}$. Further by (12), (26) and Lemma 3.2

$$\sum_{n=N_2+n_0}^{m-1} \sum_{k\in\Lambda_{n+1}^4} \bigl(l\bigl(\Gamma_n^k\bigr) - l\bigl(\Gamma_n^{k-1}\bigr)\bigr) \leqslant \frac{16(528\phi_1^{-1}+1)l(\Gamma_m^0)}{2^{M_2}-128}, \tag{29}$$

where $\Lambda_n^4 = \{k \in \{1,\dots,k_n\}\setminus\Lambda_n^2\colon N_{n-1}^{k-1}(p_n^k) \not\subset B(p_n^k, 2^{-n+M_2+1})\}$.

Choosing M_2 large enough depending on τ_0, $\phi_1 < 1$ large enough depending on M_2 and τ_0, N_2 large enough depending on M_2 and τ_0, and ε_0 small enough depending on $C_2C_3C_4$ and τ_0,

$$l\bigl(\Gamma_m^0\bigr) \leqslant (1+\tau_0)d(X) \tag{30}$$

by (24), (27)–(29).

For any $n > n_0$ let $f_n : [0,1] \to \mathcal{N}$ be $(1+\tau_0)d(X)$-Lipschitz function with $f_n([0,1]) = \Gamma_n^0$. Since each X_n is finite, we easily see by (12) that the closure of $\bigcup_{n=0}^{\infty} X_n \subset \mathcal{N}$, denoted by Y, is compact. Thus also $S(Y) := \{tz_1 + (1-t)z_2\colon z_1, z_2 \in Y,\ 0 \leqslant t \leqslant 1\}$ is compact (see [6, Lemma 5.1]). Since now $\Gamma_n^0 \subset S(Y)$ for each n, we find by the Ascoli–Arzela theorem a $(1+\tau_0)d(X)$-Lipschitz function $f : [0,1] \to \mathcal{N}$, which is the uniform limit of some subsequence of (f_n). We denote $\Gamma = f([0,1])$.

4. Size of $X\setminus\Gamma$

Let us assume for simplicity that f_n converges to f uniformly. Denote $V = X\setminus\Gamma$. In this section our goal is to show that $\mu(V) \leqslant \tau_0 d(X)$. We shall cut V into five (not necessarily disjoint) pieces, and use different arguments to show that they are small. We denote by $U(x,r)$ the closed ball in $\mathcal{N}$ with center $x \in \mathcal{N}$ and radius $r > 0$. In this section $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure on $\mathcal{N}$. Recall that

$$\mathcal{H}^1(\Gamma) \leqslant (1+\tau_0)d(X). \tag{31}$$

Notice that we can clearly assume that τ_0 is small. Set

$$V_1 = \bigl\{z \in V\colon\ \mu\bigl(B(z,r)\bigr) \leqslant \tau_0 r/20 \text{ for some } r \in \bigl[6d(z,\Gamma), d(X)\bigr]\bigr\}.$$

For each $z \in V_1$ choose $r(z) \in [6d(z,\Gamma), d(X)]$ such that $\mu(B(z,r(z))) \leqslant \tau_0 r(z)/20$ and let $w(z) \in \Gamma$ be so that $d(z,w(z)) = d(z,\Gamma)$. Denote $U_z = U(w(z), r(z)/6)$ and $5B_z = B(w(z), 5r(z)/6)$. Now $z \in U_z$ and $5B_z \subset B(z, r(z))$. By the $5r$-covering lemma we find a countable set $V_1' \subset V_1$ such that $V_1 \subset \bigcup_{z\in V_1'} 5B_z$ and the family $\{U_z\colon\ z \in V_1'\}$ is disjoint, and we get

$$\mu(V_1) \leqslant \sum_{z\in V_1'} \mu(5B_z) \leqslant \sum_{z\in V_1'} \mu\bigl(B\bigl(z, r(z)\bigr)\bigr) \leqslant \frac{\tau_0}{20}\sum_{z\in V_1'} r(z). \tag{32}$$

Assume first that Γ leaves each U_z, $z \in V_1'$. Then $\mathcal{H}^1(\Gamma \cap U_z) \geqslant r(z)/6$ for each $z \in V_1'$. Thus (32), the disjointness of the balls U_z and (31) yield

$$\mu(V_1) \leqslant \frac{3\tau_0}{10}\sum_{z\in V_1'} \mathcal{H}^1(\Gamma \cup U_z) \leqslant \frac{3\tau_0}{10}\mathcal{H}^1(\Gamma) \leqslant \frac{\tau_0 d(X)}{3}. \tag{33}$$

If $\Gamma \subset U_{z_0}$ for some $z_0 \in V_1'$ then $V_1' = \{z_0\}$ by the disjointness of the balls U_z, $z \in V_1'$, and (32) gives $\mu(V_1) \leqslant \tau_0 r(z_0)/20 \leqslant \tau_0 d(X)/20$.

Denote $H = \bigcup_{n=n_0}^{\infty} H_n$, $2B(y) = B(q_m^{-1}(y), 2^{-m(y)+4})$ for $y \in H$. We next estimate the measure of the set

$$V_2 = \bigcup_{y\in H} 2B(y).$$

By the construction the balls $U_y := U(y, 2^{-m(y)-2})$, $y \in H$, are disjoint. Assuming that Γ leaves each U_y, $y \in H$, we thus have (by (31))

$$\sum_{y\in H} 2^{-m(y)} \leqslant 4\sum_{y\in H} \mathcal{H}^1(\Gamma \cap U_y) \leqslant 4\mathcal{H}^1(\Gamma) \leqslant 5d(X).$$

If $\Gamma \subset U_{y_0}$ for some $y_0 \in H$ then $H = \{y_0\}$ by the disjointness of the balls U_y, $y \in H$, and we have

$$\sum_{y\in H} 2^{-m(y)} = 2^{-m(y_0)} \leqslant 2^{-n_0} < 2d(X).$$

Using these estimates and (9) we get

$$\mu(V_2) \leqslant \sum_{y\in H} \mu\Bigl(2B(y) \Bigm\backslash \bigcup_{z\in H_{m(y)}} B(z)\Bigr) \leqslant \sum_{y\in H} C_1\delta 2^{-m(y)} \leqslant 5C_1\delta d(X). \tag{34}$$

Now let $z \in V\backslash(V_1 \cup V_2)$. Let $n(z)$ be the integer such that

$$2^{-n(z)+M_1} \leqslant d(z,\Gamma) < 2^{-n(z)+M_1+1}. \tag{35}$$

Set $D(z) = B(z, 6d(z,\Gamma))$. If $6d(z,\Gamma) > d(X)$ then by choosing ε_1 and δ small enough depending on N_0, M_1 and μ_0 (and using Lemma 1.5) we find that $D(z) \cap \mathcal{D}_{n(z)} \neq \emptyset$. Else, since $z \notin V_1$, by choosing ε_1 small enough depending on N_0, M_1 and τ_0 and then using Lemma 1.5 we find $w \in D(z)$ such that

$$\mu\bigl(B\bigl(w, 2^{-n(z)-N_0}\bigr)\bigr) > \eta\tau_0 2^{-n(z)+M_1},$$

where $\eta > 0$ depends on N_0 and M_1. Since $z \notin V_1$ we have

$$\mu\bigl(B\bigl(w, 2^{-m}\bigr)\bigr) \geqslant \mu\bigl(B\bigl(z, 2^{-m-1}\bigr)\bigr) > \tau_0 2^{-m-1}/20$$

for all $n_0 \leqslant m \leqslant n(z) - M_1 - 5$. So by choosing δ small enough depending on N_0, M_1 and τ_0 we get that $w \in \mathcal{D}_{n(z)}$. Now $d(w, D_{n(z)}) \leqslant 2^{-n(z)+1}$ or $w \in B(y)$ for some $y \in H_{n(z)}$. In the latter case $d(z,w) > 2^{-m(y)+3}$, because $z \notin V_2$. Thus in both cases $d(w, X_{n(z)}) < 7d(z,\Gamma)$ and further $d(z, X_{n(z)}) < 13d(z,\Gamma)$. Let $y(z) \in X_{n(z)}$ be such that $d(z, y(z)) = d(z, X_{n(z)})$. By Lemma 2.1 and (35) we have

$$2^{-n(z)+M_1-1} < d\bigl(z, y(z)\bigr) < 2^{-n(z)+M_1+5}. \tag{36}$$

For $x \in X$ and $n \geqslant n_0$ we denote

$$W_n(x) = B\bigl(x, 2^{-n+M_1+6}\bigr) \cap \bigl\{z \in V\backslash(V_1 \cap V_2)\colon\ n(z) = n\bigr\}.$$

Lemma 4.1. *It holds that $\mu(W_n(x)) \leqslant \tau_0 2^{-n}/20$ for all $n \geqslant n_0$ and $x \in X$.*

Proof. Let $n \geqslant n_0$ and $x \in X$. Suppose to the contrary that $\mu(W_n(x)) > \tau_0 2^{-n}/20$. By choosing ε_1 small enough depending on N_0, M_1 and τ_0 and then using Lemma 1.5 we find $z \in W_n(x)$ such that $\mu(B(z, 2^{-n-N_0})) > \eta\tau_0 2^{-n}$, where $\eta > 0$ depends on N_0 and M_1. As before, since $z \notin V_1$ we have $\mu(B(z, 2^{-m})) > \tau_0 2^{-m}/20$ for all $n_0 \leqslant m \leqslant n - M_1 - 4$. So by choosing δ small enough depending on N_0, M_1 and τ_0 we get that $z \in \mathcal{D}_n$. Since $z \notin V_2$, we have $d(z, D_n) \leqslant 2^{-n+1}$ which contradicts (36). $\square$

Denote $V_3 = \{z \in V\backslash(V_1 \cup V_2)\colon\ N^0_{n(z)}(y(z)) \leqslant 1\}$. By (36) and Lemma 4.1

$$\mu(V_3) < \sum_{n=n_0}^{\infty} \tau_0 2^{-n+1}/20 = \tau_0 2^{-n_0+2}/20 < \tau_0 d(X)/2. \tag{37}$$

Set

$$V_4 = \{z \in V \setminus (V_1 \cup V_2)\colon \{z, v, w\} \notin \mathcal{O}(\phi_1) \text{ for some } v, w \in Z(z)\},$$

where $Z(z) = B(y(z), 2^{-n(z)+M_1}) \cap X_{n(z)}$. If $\#Z(z) \geqslant 2$ then $m_{n(z)}(v) \geqslant n(z) - M_1$ for each $v \in Z(z)$. Let $z \in V_4$ and choose $v_1, v_2 \in Z(z)$ with $\{z, v_1, v_2\} \notin \mathcal{O}(\phi_1)$. As before, by choosing N_0 large enough depending on ϕ_1 and M_1 we deduce that $c(z, w_1, w_2) \geqslant c2^{-n(z)}$ for all $w_i \in B_{n(z)}(v_i)$, $i = 1, 2$, where c is a positive constant depending on N_0, ϕ_1 and M_1. Thus

$$\int\limits_{B_{n(z)}(v_1)\times B_{n(z)}(v_2)} c(z, w_1, w_2)^2\, d\mu^2(w_1, w_2) \geqslant c^2\delta^2 2^{-2N_0}.$$

Choosing the constant K large enough depending on M_1 we have that $B_{n(z)}(v_1) \times B_{n(z)}(v_2) \subset \mathcal{T}_K(X)_z$ and further

$$\mu(V_4) \leqslant c^{-2}\delta^{-2}4^{N_0} c_K^2(X, \mu) \leqslant c^{-2}\delta^{-2}4^{N_0}\varepsilon_0 d(X). \tag{38}$$

Let now $z \in V_5$, where $V_5 = V \setminus (V_1 \cup V_2 \cup V_3 \cup V_4)$. We first show that $d(y(z), w) > 2^{-n(z)+M_1-1}$ for some $w \in N^0_{n(z)}(y(z))$. Denote $N^0_{n(z)}(y(z)) = \{u, v\}$ and assume that $\{u, v\} \subset B(y(z), 2^{-n(z)+M_1})$. Recall that $\#N^0_{n(z)}(y(z)) = 2$ since $z \notin V_3$. Now $\{u, v, y(z)\} \in \mathcal{O}(\phi_1)$ and $uy(z)v$ by Lemma 2.2 and (H1). Since $z \notin V_4$ we further have $\{z, u, v, y(z)\} \in \mathcal{O}(\phi_1)$. Choosing ϕ_1 big enough depending on M_1, assuming $d(z, u) \leqslant d(z, v)$ and using (10), (36) and Lemma 1.3 we conclude $uzy(z)v$ and $d(u, y(z)) > d(z, y(z)) > 2^{-n(z)+M_1-1}$. So in each case we may choose $u(z) \in N^0_{n(z)}(y(z))$ such that $d(u(z), y(z)) > 2^{-n(z)+M_1-1}$. By Lemma 3.2 we find Cauchy sequences $(u_n(z))_n$ and $(y_n(z))_n$ so that $\{u_n(z), y_n(z)\} \in \mathcal{E}^0_n$ and $d(u(z), u_n(z)), d(y(z), y_n(z)) < 2^{-n(z)+6}$ for all $n \geqslant n(z)$. By taking $M_1 \geqslant 9$

$$d(u_n(z), y_n(z)) > 2^{-n(z)+M_1-2} \tag{39}$$

for all $n \geqslant n(z)$.

For $n \geqslant n_0$ and $e \in \mathcal{E}^0_n$ we denote $V_5^n = \{z \in V_5\colon n(z) \leqslant n\}$ and

$$V_5^n(e) = \{z \in V_5^n\colon \{u_n(z), y_n(z)\} = e\}.$$

Lemma 4.2. *It holds that $\mu(V_5^n(e)) \leqslant \tau_0 d(a, b)/20$ for each $n \geqslant n_0$ and $e = \{a, b\} \in \mathcal{E}^0_n$.*

Proof. Let $n \geqslant n_0$ and $e = \{a, b\} \in \mathcal{E}^0_n$. By (36) and (39) for any $z \in V_5^n(e)$

$$d(z, \{a, b\}) < d(z, y(z)) + 2^{-n(z)+6} < 2^{-n(z)+M_1+6},$$
$$2^{-n(z)} < 2^{2-M_1} d(a, b).$$

Thus Lemma 4.1 gives $\mu(V_5^n(e)) \leqslant \tau_0 2^{4-M_1} d(a, b)/20$. $\square$

Lemma 4.2 and (30) now imply that $\mu(V_5^n) \leqslant \tau_0 l(\Gamma_n^0)/20 \leqslant \tau_0 d(X)/10$ for all integers $n \geqslant n_0$. Hence

$$\mu(V_5) \leqslant \tau_0 d(X)/10. \tag{40}$$

Combining (33), (34), (37), (38) and (40), choosing δ small enough depending on C_1 and τ_0, and ε_0 small enough depending on $4^{N_0}/(c\delta)^2$ and τ_0, we obtain $\mu(V) \leqslant \tau_0 d(X)$.

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